JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 29, No. 6, November–December 2006

On the Lyapunov Stability of Stationary Points Around a Central Body

A. Elipe*
Universidad de Zaragoza, 50009 Zaragoza, Spain
and
Teodoro López-Moratalla†
Real Observatorio de la Armada, 11110 San Fernando, Spain

It is well known that for a satellite about the rotating Earth, in a frame fixed in the planet, there are four stationary solutions, two of which are stable in the linear sense and the other two unstable. This result is proved by computing the eigenvalues of the linearized equations of motion. Determining the orbital stability (or Lyapunov stability) is more complicated because it requires firstly the computing of the normal form around the equilibrium and secondly the expression of this normal form in action-and-angle variables in order to apply the so-called Arnold's theorem of stability. Higher-order normal forms are necessary for some specific values. However, Arnold's theorem is useless in the presence of resonances, in which case a new technique (again related to normal forms) must be applied in order to determine the orbital stability. In this paper, we proceed symbolically, taking the harmonic coefficients as parameters. We find the Lyapunov stability diagram on the parametric plane for the stationary points. We also study resonances 2:1 and 3:1, as well as the case in which a higher-order normalization is needed. Therefore, whatever the values of the harmonic coefficients of the potential expansion are the Lyapunov stability of the stationary solutions is determined. The advantage of a symbolic analysis is that the replacement of the actual values of a planet or celestial body is sufficient to obtain the orbital stability of the stationary points. For Earth-like planets, these points are indeed stable.

I. Introduction

GEOSTATIONARY orbits are widely used, mainly in communication missions, but also in some Earth observation and certain scientific missions. Since the first launch of a spacecraft into a geostationary orbit (1963), the number of these satellites has increased up to several hundreds (e.g., see the monograph of Soop[1] and references therein).

For the Earth, assumed to be an oblate rigid body in rotation around the axis of greatest inertia, there are four equilibria, if only the zonal and tesseral harmonics up to the second order and degree are taken into account. Two of them are linearly stable and the other two unstable (e.g., see Blitzer et al.,[2] Musen and Bailie,[3] or Morando[4]).The stationary points are equilibria of the motion equations, and their linear stability is determined by the variational equations when only quadratic terms are considered in the Taylor expansion of the Hamiltonian around the equilibrium. However, linear stability, even if it is important, does not guarantee stability in a wide sense (like the Lyapunov stability) because the remaining terms in the just-mentioned expansion can destroy the stability. For the Earth and Mars, Deprit and López-Moratalla[5] and López-Moratalla[6] proved that the two linear stable points are indeed Lyapunov stable.

In recent years, missions to other celestial bodies, such as planet's satellites, asteroids, or comet nuclei, have been planned and launched. A good knowledge of the dynamics around these bodies will be of great importance to mission designs, and many studies are being carried out for bodies like Eros and Europa, targets of missions like Rosseta or Jupiter Icy Moons Orbiter, to mention just a few.

In the case of planet Earth, the analysis of geostationary points has already been done,[5,6] but the just-mentioned bodies have different characteristics; some of them are quite regular, but others are rather irregular in shape and mass distribution, and hence one could expect a different behavior of the orbits around them. This is, precisely, the main goal of this paper, to analyze the existence and Lyapunov stability of stationary points under the gravity field of a generic body (the planet) and with respect to the frame that is rotating with the planet.

In this paper, we formulate the problem from first principles. From the equations of motion—and for arbitrary values of the potential coefficients—we derive[6] the six possible equilibria (Sec. II), whereas for the particular values of the Earth there are only four stationary points. To determine their linear stability, we find the eigenvalues corresponding to the variational equations (Sec. III). Because we intend to make our analysis as general as possible, the Hamiltonian depends on two parameters, which are functions of the harmonic coefficients. Linearly unstable points are Lyapunov unstable. However, we find two positions of linearly stable equilibrium with characteristic exponents that are purely imaginary. In those cases, after the appropriate normalization[7] by the Lie transformation, the theorem of Arnold[8] about nondefinite quadratic forms is applied (Sec. IV). The latter theorem is not valid in the presence of resonances. By normalizing the Hamiltonian in extended Lissajous variables[9] and by means of an extension of the Arnold theorem,[10] the resonances of order 3 and 4 are analyzed (Sec. V), which completes the analysis of the Lyapunov stability of stationary points of a planet.

II. Equilibria

Let us consider the motion of a satellite with respect to a synodic reference frame that is rotating, as the planet does, around the axis of greatest inertia. The origin of the reference frame coincides with the center of mass of the planet, and the axes are the principal axes of inertia (z the axis of greatest inertia and x the axis of minor inertia). Note that this is not the usual way[11] of selecting the axis; thus, the harmonic coefficients $\Gamma_{\ell,m}$, $\Delta_{\ell,m}$, do not coincide with the classical ones (of the geopotential) $C_{\ell,m}$, $S_{\ell,m}$. We consider the satellite as a mass point and take until the second order and

Presented as Paper 2004-5303 at the AIAA/AAS Astrodynamics Specialist Conference, Providence, RI, 16–19 August 2004; received 9 April 2005; revision received 2 August 2005; accepted for publication 4 August 2005.

*Professor, Grupo de Mecánica Espacial. Associate Fellow AIAA.
†Commander, Head of Ephemerides Section.

Table 1 Harmonic coefficients for the Earth and Mars for the principal axes of inertia

Harmonics	Earth	Mars
$\Gamma_{2,0}$	$-0.1082630 \times 10^{-2}$	$-0.1958616 \times 10^{-2}$
$\Gamma_{2,1}$	0	0
$\Gamma_{2,2}$	0.1814964×10^{-5}	0.1516035×10^{-3}
$\Delta_{2,1}$	0	0
$\Delta_{2,1}$	0	0

degree in the potential expansion. Our harmonic coefficients are related to the classical ones by the expressions $\Gamma_{2,1} = \Delta_{2,1} = \Delta_{2,2} = 0$, $\Gamma_{2,0} = C_{2,0}$, $\Gamma_{2,2} = \sqrt{(C_{2,2}^2 + S_{2,2}^2)}$, which shows the advantages of this choice. In Table 1 we give the values of the harmonic coefficients for the Earth and Mars, derived from WGS-84 and Mars50c, respectively.

We also suppose that the planet rotates about the z axis with constant velocity ν. Under these assumptions, the Hamiltonian in the synodic frame (e.g., see Ref. 12) is

$$\mathcal{H} = \tfrac{1}{2}(X^2 + Y^2 + Z^2) - \nu(xY - yX) + \mathcal{V}(x, y, z) \tag{1}$$

with potential $\mathcal{V}(x, y, z)$ defined by

$$\mathcal{V} = -\frac{\mu}{r}\left\{1 + \left(\frac{\oplus}{r}\right)^2\left[3\Gamma_{2,2}\frac{x^2 - y^2}{r^2} - \frac{1}{2}\Gamma_{2,0}\left(1 - 3\frac{z^2}{r^2}\right)\right]\right\} \tag{2}$$

where μ is the Gaussian constant, $r = \sqrt{(x^2 + y^2 + z^2)}$ is the radial distance of the satellite, $\oplus$ the planet's equatorial radius, and where the harmonic coefficients are such that $\Gamma_{2,0} < 0 < \Gamma_{2,2}$ because of the planetary oblateness and our choice of the axes.

The equations of motion corresponding to the preceding Hamiltonian are

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial X} = X + \omega y, \qquad \dot{y} = \frac{\partial\mathcal{H}}{\partial Y} = Y - \omega x, \qquad \dot{z} = \frac{\partial\mathcal{H}}{\partial Z} = Z$$

$$\dot{X} = -\frac{\partial\mathcal{H}}{\partial x} = \omega Y - \mu\frac{x}{r^3}\left\{1 - 3\frac{\oplus^2}{r^2}\left[\frac{1}{2}\Gamma_{2,0}\left(1 - 5\frac{z^2}{r^2}\right) + \Gamma_{2,2}\left(2 - 5\frac{x^2 - y^2}{r^2}\right)\right]\right\}$$

$$\dot{Y} = -\frac{\partial\mathcal{H}}{\partial y} = -\omega X - \mu\frac{y}{r^3}\left\{1 - 3\frac{\oplus^2}{r^2}\left[\frac{1}{2}\Gamma_{2,0}\left(1 - 5\frac{z^2}{r^2}\right) - \Gamma_{2,2}\left(2 + 5\frac{x^2 - y^2}{r^2}\right)\right]\right\}$$

$$\dot{Z} = -\frac{\partial\mathcal{H}}{\partial z} = -\mu\frac{z}{r^3}\left\{1 - 3\frac{\oplus^2}{r^2}\left[\frac{1}{2}\Gamma_{2,0}\left(3 - 5\frac{z^2}{r^2}\right) - 5\Gamma_{2,2}\frac{x^2 - y^2}{r^2}\right]\right\} \tag{3}$$

Equilibria are found by zeroing the preceding equations. Although there are solutions for $z \neq 0$, from here on we will restrict our analysis to the equatorial case ($z = 0$, $Z = 0$), thus the two-degrees-of-freedom problem and

$$\begin{cases} y = 0 \\ \text{and} \\ (r/a_k)^5 - (r/a_k)^2 = 3\left(-\frac{1}{2}\Gamma_{2,0} + 3\Gamma_{2,2}\right)(\oplus/a_k)^2 \end{cases} \tag{4}$$

or

$$\begin{cases} x = 0 \\ \text{and} \\ (r/a_k)^5 - (r/a_k)^2 = 3\left(-\frac{1}{2}\Gamma_{2,0} - 3\Gamma_{2,2}\right)(\oplus/a_k)^2 \end{cases} \tag{5}$$

where the scaling factor $a_k = (\mu/\nu^2)^{1/3}$ is the semimajor axis of a Keplerian orbit with mean motion $n = \nu$.

The first case, Eq. (4), has only one positive real solution r_1, which is $r_1 \geq a_k$. The corresponding implicit equation can be solved by Newton–Raphson, starting from $r = a_k$. One simple iteration results accurate to the order of ϵ^4, while two iterations are sufficient up to $\mathcal{O}(\epsilon)$ (Ref. 8):

$$r/a_k = 1 + \epsilon - 3\epsilon^2 + (44/3)\epsilon^3 - (260/3)\epsilon^4 + 567\epsilon^5 - (35581/9)\epsilon^6 + (259160/9)\epsilon^7 + \mathcal{O}(\epsilon)^8 \tag{6}$$

where parameter ϵ is

$$\epsilon = \epsilon_x = \left(-\tfrac{1}{2}\Gamma_{2,0} + 3\Gamma_{2,2}\right)(\oplus/a_k)^2 > 0 \tag{7}$$

In the second case, Eq. (5), by setting $\epsilon = \epsilon_y$, with

$$\epsilon_y = \left(-\tfrac{1}{2}\Gamma_{2,0} - 3\Gamma_{2,2}\right)(\oplus/a_k)^2 \tag{8}$$

Eq. (6) is applicable again. Note that now ϵ_y can have any sign. For $\epsilon_y \geq 0$ (which happens in an Earth-like planet), Eq. (5) has only one real and positive root ($r_2 \geq a_k$). However, when $\epsilon_y < 0$, there are either none or two (r_2', r_2) positive real solutions of Eq. (5), and they are such that

$$0 < r_2' \leq \left(\tfrac{2}{5}\right)^{\frac{1}{3}} a_k \leq r_2 < a_k$$

This case has been studied in detail by Howard.[13]

In summary, the possible equilibria of Eq. (6) are points on xy plane, placed either on the x axis, $E_1(\pm r_1, 0)$, or on the y axis, $E_2(0, \pm r_2)$ and $E_2'(0, \pm r_2')$.

III. Hamiltonian in a Neighborhood of the Equilibrium

We make a symplectic transformation $\boldsymbol{x}(x, y, X, Y) \longmapsto \boldsymbol{\zeta}_j(\xi_j, \eta_j, \Xi_j, H_j)$, with $j = \{1, 2\}$ to put the equilibrium at the origin. This is achieved by means of

$$x = \eta_1 + r_1, \qquad X = H_1, \qquad y = -\xi_1, \qquad Y = \nu r_1 - \Xi_1$$

for the point E_1, and

$$x = \xi_2, \qquad X = \Xi_2 - \nu r_2, \qquad y = \eta_2 + r_2, \qquad Y = H_2$$

for E_2 (for E_2' we change r_2 by r_2').

The transformed Hamiltonians are

$$\mathcal{H}^{(j)} = -\tfrac{1}{2}\nu^2 r_j^2 + \tfrac{1}{2}\left(\Xi_j^2 + H_j^2\right) - \nu(\xi_j H_j - \eta_j \Xi_j) - \nu^2 r_j \eta_j - (\mu/\rho_j)\Big(1 - \left(\oplus^2/\rho_j^2\right)\left\{\tfrac{1}{2}\Gamma_{20} - (-1)^j 3\Gamma_{22}\left[\left(\xi_j^2 - \eta_j^2 - 2r_j\eta_j - r_j^2\right)/\rho_j^2\right]\right\}\Big)$$

where ρ_j is given by

$$\rho_j = \sqrt{\xi_j^2 + (\eta_j + r_j)^2}, \qquad (j = 1, 2)$$

By expanding this Hamiltonian in a Taylor series around the origin (the equilibrium in this case) and neglecting constant terms, we get

$$\mathcal{H}^{(j)} = \mathcal{H}_2^{(j)} + \mathcal{H}_3^{(j)} + \mathcal{H}_4^{(j)} + \cdots \tag{9}$$

where each term $\mathcal{H}_n^{(j)}$ is a homogeneous polynomial of degree n in the new variables.

The quadratic Hamiltonian is

$$\mathcal{H}_2^{(j)} = \tfrac{1}{2}\left(\Xi_j^2 + H_j^2\right) - \nu(\xi_j H_j - \eta_j \Xi_j) + \tfrac{1}{2}\nu^2\left(\alpha_j \xi_j^2 + \beta_j \eta_j^2\right) \tag{10}$$

with parameters α_j, β_j

$$\alpha_j = 1 - 12(-1)^j \Gamma_{22}\left(a_k^3 \oplus^2 / r_j^5\right) \qquad \text{and} \qquad \beta_j = 2\left(a_k^3 / r_j^3 - 2\right) \tag{11}$$

For the sake of simplifying the notation, from here on we will drop the script j that refers to each one of the equilibria.

The rest of the terms of expansion 9, until order 7, are

$$\mathcal{H}_3 = \nu^2\left(c_{21}\xi^2\eta + c_{03}\eta^3\right)$$

$$\mathcal{H}_4 = \nu^2\left(c_{40}\xi^4 + c_{22}\xi^2\eta^2 + c_{04}\eta^4\right)$$

$$\mathcal{H}_5 = \nu^2\left(c_{41}\xi^4\eta + c_{23}\xi^2\eta^3 + c_{05}\eta^5\right)$$

$$\mathcal{H}_6 = \nu^2\left(c_{60}\xi^6 + c_{42}\xi^4\eta^2 + c_{24}\xi^2\eta^4 + c_{06}\eta^6\right) \tag{12}$$

The dimensionless coefficients c_{ij} depend on the position of the equilibrium and are given by

$\mathcal{H}_3$:

$$rc_{21} = (4 - 5\alpha + \beta)/2, \qquad rc_{03} = -(8 + 7\beta)/6$$

$\mathcal{H}_4$:

$$r^2 c_{40} = (9 - 10\alpha + \beta)/8, \qquad r^2 c_{22} = 3(-12 + 10\alpha - 3\beta)/4$$

$$r^2 c_{04} = 3 + 2\beta$$

$\mathcal{H}_5$:

$$r^3 c_{41} = -5(15 - 14\alpha + 2\beta)/8, \qquad r^3 c_{23} = 5(20 - 14\alpha + 5\beta)/4$$

$$r^3 c_{05} = -5 - 3\beta$$

$\mathcal{H}_6$:

$$r^4 c_{60} = -5(22 - 21\alpha + 2\beta)/48$$

$$r^4 c_{42} = 5(132 - 112\alpha + 19\beta)/16$$

$$r^4 c_{24} = -5(44 - 28\alpha + 11\beta)/4, \qquad r^4 c_{06} = (44 + 25\beta)/6 \tag{13}$$

Let us turn back to the term $\mathcal{H}_2$ that gives rise to the linearized equations of motion. Indeed, if J denotes the standard symplectic matrix and A the matrix of the quadratic form $\mathcal{H}_2$, the variational equations are

$$\dot{\zeta} = JA\zeta = B\zeta$$

with J, A, and B the matrices

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} \nu^2\alpha & 0 & 0 & -\nu \\ 0 & \nu^2\beta & \nu & 0 \\ 0 & \nu & 1 & 0 \\ -\nu & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & \nu & 1 & 0 \\ -\nu & 0 & 0 & 1 \\ -\nu^2\alpha & 0 & 0 & \nu \\ 0 & -\nu^2\beta & -\nu & 0 \end{pmatrix}$$

The eigenvalues of B, the roots of the characteristic equation

$$\det(\lambda I - B) = \lambda^4 + \nu^2(\alpha + \beta + 2)\lambda^2 + \nu^4(1 - \alpha)(1 - \beta) = 0$$

determine the linear stability. Because the system is Hamiltonian, to obtain linear stability, it is necessary and sufficient that all of the eigenvalues be purely imaginary and that matrix B be diagonalizable. Thus, if we denote by

$$\omega_1^2 = (\nu^2/2)\left[-(\alpha + \beta + 2) + \sqrt{(\alpha - \beta)^2 + 8(\alpha + \beta)}\right]$$

$$\omega_2^2 = (\nu^2/2)\left[-(\alpha + \beta + 2) - \sqrt{(\alpha - \beta)^2 + 8(\alpha + \beta)}\right]$$

and if $\omega_1^2, \omega_1^2 \le 0$, we have linear stability when the eigenvalues $\pm i\omega_1$, $\pm i\omega_2$, are such that $\omega_1 \ne \omega_2$ and $\omega_1, \omega_2 > 0$. If $\omega_1 = \omega_2$ or $\omega_j = 0$, it is necessary that matrix B be diagonalizable too.

On the parameter plane α, β, there are two regions in which the preceding conditions are fulfilled, namely,

$$R_I = \{(\alpha, \beta) \in R^2 | \alpha, \beta > 1\}$$

and

$$R_{II} = \{(\alpha, \beta) \in R^2 | -3 < \alpha, \beta < 1$$
$$\text{and} \qquad (\alpha - \beta)^2 + 8(\alpha + \beta) > 0\} \tag{14}$$

On the boundaries of these regions, the four eigenvalues are either $(\pm\omega_1, \pm\omega_2)$, or $(0, 0, \pm\omega_2)$, in other words, multiple eigenvalues. Point $\alpha = \beta = 1$ is the only one which satisfies that matrix B is diagonalizable; hence, it is the only point on the boundary at which the equilibrium is linearly stable.

At the equilibrium E_1, parameters (11) are

$$\alpha_1 = 1 + 12\Gamma_{22}\left(a_k^3 \oplus^2 / r_1^5\right) > 1 \qquad \text{and} \qquad \beta_1 = 2\left(a_k^3 / r_1^3 - 2\right) < 1$$

because $\Gamma_{22} > 0$ and $r_1 > a_k$. Consequently, the equilibrium is linearly unstable, and thus, unstable.

At point E_2', we have

$$\alpha_2' = 1 - 12\Gamma_{22}\left(a_k^3 \oplus^2 / r_2'^5\right) < 1 \qquad \text{and} \qquad \beta_2' = 2\left(a_k^3 / r_2'^3 - 2\right)$$

but now, we recall that $0 < r_2' \le (2/5)^{1/3} a_k < a_k$; thus, $\beta_2' > 1$, and, therefore, this point is unstable.

Finally, for point E_2 the analysis is a little more complex because now r_2 is such that $0 < r_2' \le (2/5)^{1/3} a_k \le r_2 < a_k$ and $\alpha_2, \beta_2 < 1$. To see its stability, we have to check whether the second condition of region R_{II} is satisfied. After some algebra,[6] and taking into account that r_2 depends also on harmonics $\Gamma_{2,0}$ and $\Gamma_{2,2}$ through the quintic equation (5), it can be proved that for an Earth-like planet both parameters are indeed on region R_{II}, so that the point is linearly stable. However, there are values of harmonics $\Gamma_{2,0}$ and $\Gamma_{2,2}$ for which the point is unstable. For details, see López-Moratalla.[6]

As an illustration of bodies which equilibria and stability differ from the Earth case, we consider triaxial ellipsoids with data in Table 2. Units have been chosen such that $\nu = 1$, $\mu = 1$, and, hence, $a_k = 1$.

Table 2 Coefficients for two triaxial rigid bodies of semi-axes ($a = \oplus, b, c$)[a]

Coefficients	Body 1	Body 2
(a, b, c)	(1/3, 1/6, 1/7)	(1/4, 1/8, 1/9)
$\Gamma_{2,0}$	−0.176531	−0.170988
$\Gamma_{2,2}$	0.075	0.075
r_2'	0.214553 (U)	0.162078 (U)
r_2	0.984058 (U)	0.991043 (S)
α_2	0.891633	0.941162
β_2	−1.90122	−1.94528
Δ_2	−0.276661	0.298602

[a]For these two bodies there are four equilibria on the y axis. Points $E_2'(0, \pm r_2')$ are always unstable, whereas $E_2(0, \pm r_2)$ are unstable for body 1 and stable for body 2, according to the values of parameters α, β, and $\Delta = (\alpha - \beta)^2 + 8(\alpha + \beta)$.

IV. Lyapunov's Stability

Linear stability does not imply Lyapunov stability. Indeed, the quadratic part (10) of the original Hamiltonian is an approximation, and the remaining terms could destroy the stable behavior of the truncated Hamiltonian at order 2. There are several ways to solve this problem. One possible answer is to find a Lyapunov function, which, in most cases, is rather difficult. Because the Hamiltonian $\mathcal{H}_2$ is a quadratic form, another method consists of checking its sign. If it were sign defined, we would ensure Lyapunov stability by Diritchlet's theorem (e.g., see Ref. 14) because the Hamiltonian is an integral.

In this case, the quadratic Hamiltonian is a positive-defined inside region R_I of Eq. (14), while it is nondefined in region R_{II} of Eq. (14). Unfortunately, the unique linearly stable equilibrium lies on R_{II}, and Diritchlet's theorem cannot be applied. However, for this situation and for two degrees of freedom Arnold's theorem will help us to decide the Lyapunov stability. This theorem reads as follows:

Theorem 1 (Arnold): Consider a two-degrees-of-freedom Hamiltonian system $\mathcal{H}$ expressed, in the real canonical coordinates $(\Phi_1, \Phi_2, \phi_1, \phi_2)$ as

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \cdots + \mathcal{H}_{2n} + \tilde{\mathcal{H}}$$

where the following is true:

1) $\mathcal{H}$ is real analytic in a neighborhood of the origin $\boldsymbol{R}^4$.

2) $\mathcal{H}_{2k}$, $1 \le k \le n$, is a homogeneous polynomial of degree k in Φ_i, with real coefficients. In particular,

$$\mathcal{H}_2 = \omega_1\Phi_1 - \omega_2\Phi_2, \qquad 0 < \omega_1, \qquad 0 < \omega_2$$
$$\mathcal{H}_4 = \tfrac{1}{2}\left(A\Phi_1^2 - 2B\Phi_1\Phi_2 + C\Phi_2^2\right)$$

3) $\tilde{\mathcal{H}}$ has a power expansion in Φ_i, which starts with terms at least of order $2n+1$.

Under these assumptions, the origin is a stable equilibrium provided for some k, $2 \le k \le n$, $\mathcal{H}_2$ does not divide $\mathcal{H}_{2k}$ or equivalently, provided $D_{2k} = \mathcal{H}_{2k}(\omega_2, \omega_1) \neq 0$, and for $2 \le j < kD_{2j} = \mathcal{H}_{2j}(\omega_2, \omega_1) = 0$.

This theorem assumes that the Hamiltonian is expressed in its normal form and that there is no commensurability among the natural frequencies ω_1 and ω_2 of the principal part. Thus, in order to apply the theorem we have to take the following steps: first, to express Hamiltonian (9) (evaluated at the equilibrium) in action-and-angle variables $(\Phi_1, \Phi_2, \phi_1, \phi_2)$ and have the quadratic part in diagonal form; second, to normalize the transformed Hamiltonian; and third, to compute the discriminant D_4 and determine when it is null. For the regions in which $D_4 = 0$, we need to carry out the normalization up to a higher order.

Thus we are interested in points (E_2) in region R_{II} of Eq. (14), which are linearly stable. For these points, the four eigenvalues of matrix A of Hamiltonian (10) are complex and different from each other. Besides, this matrix is Hamiltonian, that is, $A = JA^TJ$, which guarantees[15] that there is a canonical transformation to an eigenvectors basis in which the quadratic form (10) is diagonal.

Following Deprit,[16] we define a symplectic transformation $\boldsymbol{w} = (u, v, U, V) \longmapsto \boldsymbol{\zeta} = (\xi, \eta, \Xi, H)$ by the linear transformation $\boldsymbol{\zeta} = \mathcal{B}\boldsymbol{w}$, where $\mathcal{B}$ is the matrix

$$\mathcal{B} = \begin{pmatrix} ia_1 & -ia_2 & a_1 & a_2 \\ -b_1 & b_2 & -ib_1 & -ib_2 \\ b_1\nu - a_1\omega_1 & a_2\omega_2 - b_2\nu & -i(a_1\omega_1 - b_1\nu) & -i(a_2\omega_2 - b_2\nu) \\ i(a_1\nu - b_1\omega_1) & -i(a_2\nu - b_2\omega_2) & a_1\nu - b_1\omega_1 & a_2\nu - b_2\omega_2 \end{pmatrix} \tag{15}$$

and coefficients a_1, a_2, b_1, b_2 are given by the expressions

$$a_1^2 = \frac{\omega_1^2 + \nu^2(1-\beta)}{2\omega_1\left(\omega_1^2 - \omega_2^2\right)}, \qquad b_1^2 = \frac{\omega_1^2 + \nu^2(1-\alpha)}{2\omega_1\left(\omega_1^2 - \omega_2^2\right)}$$
$$a_2^2 = \frac{\omega_2^2 + \nu^2(1-\beta)}{2\omega_2\left(\omega_1^2 - \omega_2^2\right)}, \qquad b_2^2 = \frac{\omega_2^2 + \nu^2(1-\alpha)}{2\omega_2\left(\omega_1^2 - \omega_2^2\right)}$$

After extensive symbolic algebra, it can be proved that Eq. (10) is converted into

$$\mathcal{H}_2 = i\omega_1 uU + i\omega_2 vV \tag{16}$$

and each term $\mathcal{H}_k$ of Hamiltonian (9) is a homogeneous polynomial of degree k in the variables $\boldsymbol{w} = (u, v, U, V)$.

Using Poincaré variables $(\phi_1, \phi_2, \Phi_1, \Phi_2)$, related to the complex variables just defined by the symplectic transformation

$$u = \sqrt{\Phi_1}\exp(i\phi_1), \qquad v = \sqrt{\Phi_2}\exp(-i\phi_2)$$
$$U = -i\sqrt{\Phi_1}\exp(-i\phi_1), \qquad V = i\sqrt{\Phi_2}\exp(i\phi_2) \tag{17}$$

the linearized Hamiltonian (16) becomes

$$\mathcal{H}_2 = \omega_1\Phi_1 - \omega_2\Phi_2$$

precisely in the form required by Arnold's theorem. Thus, we normalize the Hamiltonian.

The composition of the two symplectic transformations (15) and (17) can be explicitly written as

$$\xi = -2a_1\sqrt{\Phi_1}\sin\phi_1 - 2a_2\sqrt{\Phi_2}\sin\phi_2$$
$$\eta = -2b_1\sqrt{\Phi_1}\cos\phi_1 + 2b_2\sqrt{\Phi_2}\cos\phi_2$$
$$\Xi = -2(a_1\omega_1 - \nu b_1)\sqrt{\Phi_1}\cos\phi_1 + 2(a_2\omega_2 - \nu b_2)\sqrt{\Phi_2}\cos\phi_2$$
$$H = -2(a_1\omega_1 - b_1\omega_1)\sqrt{\Phi_1}\sin\phi_1 - 2(a_2\nu - b_2\omega_2)\sqrt{\Phi_2}\cos\phi_2 \tag{18}$$

A. Normal Form

The Lie derivative associated with $\mathcal{H}_0$ is the partial differential operator

$$L_0 : F \to (F, \mathcal{H}_0)$$

mapping F onto its Poisson bracket to the right with $\mathcal{H}_0$. The kernel of L_0 is the set of functions F such that $L_0(F) = 0$; the image of L_0 is the set of functions F of the form $F = L_0(G)$.

Normalization of a Hamiltonian of type

$$\mathcal{H}(\boldsymbol{p}, \boldsymbol{P}, \epsilon) = \sum_{n\ge 0}\epsilon^n\mathcal{H}_n(\boldsymbol{p}, \boldsymbol{P})$$

we recall[17] is a one-parameter family of canonical transformations $\varphi : (\boldsymbol{p}', \boldsymbol{P}', \epsilon) \to (\boldsymbol{p}, \boldsymbol{P})$, which changes $\mathcal{H}$ into a function $\varphi^{\#}\mathcal{H}(\boldsymbol{p}', \boldsymbol{P}', \epsilon) = \mathcal{H}\,[\boldsymbol{p}(\boldsymbol{p}', \boldsymbol{P}', \epsilon), \boldsymbol{P}(\boldsymbol{p}', \boldsymbol{P}', \epsilon), \epsilon]$ in the kernel of L_0.

In this case, we normalize with respect to the Hamiltonian $\mathcal{H}_2$. Instead of using the Poincaré variables, we will use the complex ones because in this case the Lie derivative associated with $\mathcal{H}_2$ is the operator

$$L_2 = i\omega_1\left(u\frac{\partial}{\partial u} - U\frac{\partial}{\partial U}\right) + i\omega_2\left(v\frac{\partial}{\partial v} - V\frac{\partial}{\partial V}\right)$$

In the algebra of homogeneous polynomials in (u, v, U, V),

$$L_2(u^mU^nv^pV^q) = [i\omega_1(m-n) + i\omega_2(p-q)]u^mU^nv^pV^q$$

in other words, the monomial $u^mU^nv^pV^q$ is an eigenvector of the Lie derivative, and its eigenvalue is $i\omega_1(m-n) + i\omega_2(p-q)$. The kernel of L_2 is generated by monomials of type $(uU)^m(vV)^p$; hence, the transformed terms $\varphi^{\#}\mathcal{H}_{2k+1} = 0$ for all $k > 1$.

The normalization up to the fourth order has been carried out symbolically in complex variables and expressed in Poincaré's variables.

The normalized Hamiltonian takes the form

$$\varphi^{\#}\mathcal{H} = \omega_1\Phi_1 - \omega_2\Phi_2 + A\Phi_1^2 - 2B\Phi_1\Phi_2 + C\Phi_2^2 + \mathcal{H}_6 + \cdots$$

where the coefficients A, B, and C depend on the structural parameters. Thus, we are in the hypothesis of Arnold's theorem, and we only have to check whether the discriminant

$$D_4 = \mathcal{H}_4(\omega_2, \omega_1) = A\omega_2^2 - 2B\omega_1\omega_2 + C\omega_1^2$$

is zero or not. If it is not zero, the equilibrium is stable in the Lyapunov sense; if it is zero, it will be necessary to go further in the normalization and compute $\mathcal{H}_6$ and then the discriminant $D_6 = \mathcal{H}_6(\omega_2, \omega_1)$; and check again whether it is null or not, and so on.

In terms of the frequencies ω_1 and ω_2, the discriminant D_4 is

$$D_4 = (M - RP)/Q \tag{19}$$

where Q, R, M, and P are

$$Q = 48\omega_1^4\omega_2^4\left(\omega_1^2 - 4\omega_2^2\right)\left(4\omega_1^2 - \omega_2^2\right)\left(\omega_1^2 - \omega_2^2\right)^2 r_2^2$$

$$R^2 = \left(\omega_1^2 - \omega_2^2\right)^2 + 8\omega^2\left(2\omega^2 - \omega_1^2 - \omega_2^2\right)$$

$$M = \sum_{0\le\ell\le5}\ \sum_{\substack{0\le m\le n\le7\\ m+n=9-\ell}} M_{\ell,m,n}\nu^{2\ell}\left(\omega_1^{2m}\omega_2^{2n} + \omega_1^{2n}\omega_2^{2m}\right)$$

$$P = \sum_{0\le\ell\le4}\ \sum_{\substack{0\le m\le n\le6\\ m+n=8-\ell}} P_{\ell,m,n}\nu^{2\ell}\left(\omega_1^{2m}\omega_2^{2n} + \omega_1^{2n}\omega_2^{2m}\right)$$

where the coefficients $M_{\ell,m,n}$, $P_{\ell,m,n}$ are given in Table 3 (first and second columns).

Note that the denominator Q vanishes when there is commensurability among the frequencies ($\omega_1 = 2\omega_2$ and $\omega_1 = \omega_2$), but these are cases of resonances so far excluded: they will be analyzed in the next section. Note also that resonance 1:1 does not need to be analyzed because this resonance is the limit of the linear stability–instability, and precisely at this value of the parameters the matrix of the quadratic form is not diagonalizable and thus the equilibrium is unstable.

Because we presented Hamiltonian (9) in terms of parameters α and β, we also give the expression of discriminant D_4 as function of α and β, that is,

$$D_4 = [N/24(1-\beta)^2 D]\left(\nu^2/r_2^2\right) \tag{20}$$

In this expression, the coefficients N and D are the polynomials

$$N = \sum_{0\le m\le5}\ \sum_{0\le n\le7-m} A_{m,n}\alpha^m\beta^n \tag{21}$$

$$D = [(\alpha-\beta)^2 + 8(\alpha+\beta)][4(\alpha-\beta)^2 - 9(1+\alpha\beta) + 41(\alpha+\beta)] \tag{22}$$

and coefficients $A_{m,n}$ are listed in Table 3 (third column).

In Fig. 1 we plot function D_4 as a function of the harmonic Γ_{22}. We fix harmonic Γ_{20} corresponding to the Earth, and the units of time and length are taken $\mu = 1$, $\nu = 1$, and $a_k = 1$. We see two asymptotic lines: the one to the right corresponding to resonance 1:1, and the one to the left corresponding to resonance 2:1. Note that because the normalization is carried out to second order, we

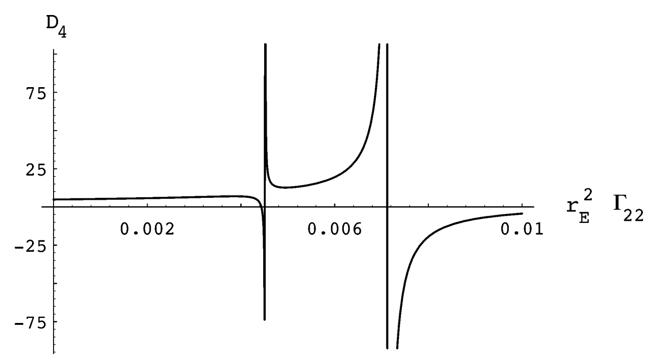

Fig. 1 Evolution of discriminant D_4 vs harmonic Γ_{22}, when the other harmonic Γ_{20} is taken fixed (the one of the Earth). Units of time and length are chosen so that $\mu = 1$ and $\nu = 1$. The first discontinuity on the left corresponds is to resonance 2:1; the second discontinuity corresponds to resonance 1:1, which is the limit for the linear stability. Points to the right of this value are linearly unstable. We can also see that just before the first discontinuity there is value of Γ_{22} in which $D_4 = 0$.

Table 3 Coefficients in discriminant D_4

M	P	A		
$M_{0,2,7} = -212$	——		——	
$M_{0,3,6} = 1086$	——		——	
$M_{0,4,5} = -874$	——		——	
	$P_{0,2,6} = 116$		——	
$M_{1,1,7} = 840$	$P_{0,3,5} = -770$	$A_{0,0} = -57816$	$A_{1,0} = 191232$	$A_{2,0} = 31175$
$M_{1,2,6} = -5842$	$P_{0,4,4} = 654$	$A_{0,1} = 130032$	$A_{1,1} = 105514$	$A_{2,1} = -92485$
$M_{1,3,5} = -9500$		$A_{0,2} = 188207$	$A_{1,2} = -60780$	$A_{2,2} = -195053$
$M_{1,4,4} = 1542$	$P_{1,1,6} = -840$	$A_{0,3} = 88807$	$A_{1,3} = -93886$	$A_{2,3} = 7153$
$M_{2,0,7} = -4500$	$P_{1,2,5} = 5850$	$A_{0,4} = 15504$	$A_{1,4} = -18724$	$A_{2,4} = 5608$
$M_{2,1,6} = -195$	$P_{1,3,4} = 12414$	$A_{0,5} = 346$	$A_{1,5} = -1956$	$A_{2,5} = 602$
$M_{2,2,5} = 140310$	$P_{2,0,6} = 4500$	$A_{0,6} = -532$	$A_{1,6} = 100$	——
$M_{2,3,4} = 161889$	$P_{2,1,5} = 1335$	$A_{0,7} = -48$	——	——
	$P_{2,2,4} = -122295$			
$M_{3,0,6} = 54000$	$P_{2,3,3} = -102780$	$A_{3,0} = -80454$	$A_{4,0} = -74745$	$A_{5,0} = -5504$
$M_{3,1,5} = -32460$		$A_{3,1} = -186420$	$A_{4,1} = 35711$	$A_{5,1} = 1168$
$M_{3,2,4} = -1158780$	$P_{3,0,5} = -36000$	$A_{3,2} = 77946$	$A_{4,2} = -11452$	$A_{5,2} = -164$
$M_{3,3,3} = -889920$	$P_{3,1,4} = 37800$	$A_{3,3} = 1404$	$A_{4,3} = 986$	——
	$P_{3,2,3} = 727200$	$A_{3,4} = -1476$	——	——
$M_{4,0,5} = -216000$			——	
$M_{4,1,4} = 270000$	$P_{4,0,4} = 72000$		——	
$M_{4,2,3} = 4093200$	$P_{4,1,3} = -190800$		——	
	$P_{4,2,2} = -496800$			
$M_{5,0,4} = 288000$				
$M_{5,1,3} = -763200$				
$M_{5,2,2} = -1987200$				

First two columns, coefficients of polynomials M and P in discriminant D_4 as function of frequencies ω_1 and ω_2 [Eq. (19)]. Last three columns, coefficients of polynomial N in expression (20) of discriminant D_4 in terms of parameters α and β.

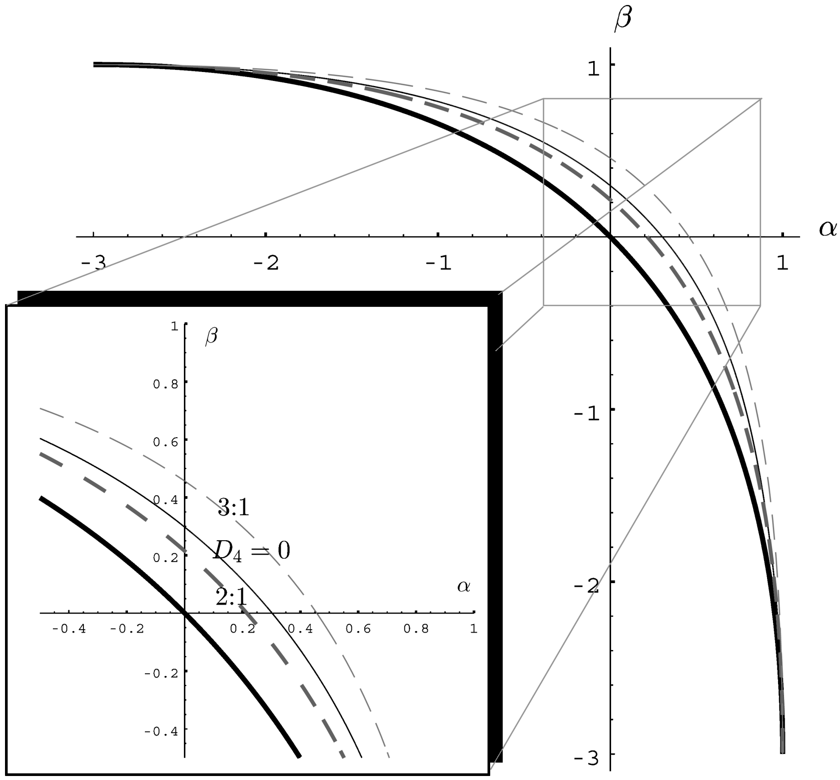

Fig. 2 Bifurcations on parametric plane α and β: ▬▬, limit of the linear stability [region R_{II}, Eq. (14)], which coincides with 1:1 resonance; ——, points in which discriminant $D_4 = 0$; ▬ ▬ ▬, we plot the points of resonance 2:1; and finally, in - - - -, we see resonance 3:1. The curves corresponding to resonance 2:1, and $D_4 = 0$ cut each other at point $\alpha = 0.982358864$, $\beta = -2.372564503$.

are dealing with monomials of fourth degree in ζ; hence, there are resonant terms to fourth order either in the Hamiltonian or in the generating function. In the present case, the resonance 3:1 appears in the latter, whereas the low-order resonances are in the normalized Hamiltonian. Proceeding with the normalization to higher orders, we will face other resonances.

If we replace the value of Γ_{22} for the Earth, we find $D_4 = 1.49991 \neq 0$. Consequently, we can conclude that for the Earth, point E_2 is Lyapunov stable.

We note also in the figure that there is a value at which discriminant D_4 vanishes. It happens for $\oplus^2\Gamma_{22} = 4.3862 \times 10^{-3}$ ($\Gamma_{22} = 0.192385$). For this particular case, if we want to apply Arnold's theorem we have to carry out the normalization two orders higher and see whether D_6 is null.

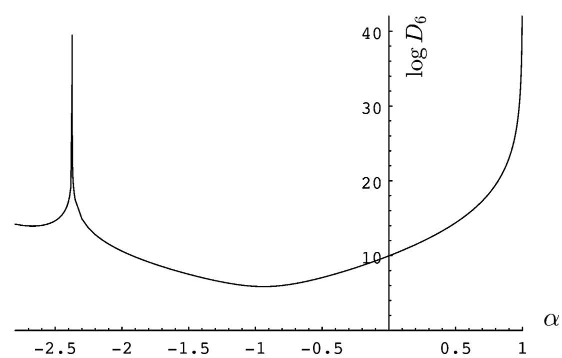

Fig. 3 Evolution of $\log D_6$ vs α along curve $D_4 = 0$. We see that $\log D_6 > 1$, then $D_6 > 0$, and the equilibrium is Lyapunov stable.

B. Case $D_4 = 0$

For the harmonic $\Gamma_{2,0}$ of the Earth, there is a certain value of $\Gamma_{2,2}$ at which $D_4 = 0$. Because we want to make the analysis as general as possible, we proceed to compute the values at which discriminant $D_4 = 0$ on parameter plane α and β. But because we already had the expression of D_4 in terms of α and β in Eq. (20), it is sufficient to find the roots of $N = 0$ from Eq. (21) in the range $-3 < \alpha, \beta < 0$, with the coefficients given in Table 3. The result appears as the thick dashed curve in Fig. 2. For every point on this curve, the discriminant is $D_4 = 0$; then, to check its Lyapunov stability using Arnold's theorem we must go further with the normalization, and compute the determinant $D_6 = \mathcal{H}_6(\omega_2, \omega_1)$ and check whether this function is zero or not. If not, the equilibrium is Lyapunov stable.

By computing for each pair (α, β) belonging to curve $D_4 = 0$ the normalization until the term $\mathcal{H}_6(\Phi_1, \Phi_2)$ and then replacing $(\Phi_1, \Phi_2) \mapsto (\omega_2, \omega_1)$, we obtain the function $D_6(\alpha, \beta)$. In Fig. 3 we plot $\log D_6$ vs α along the curve $D_4 = 0$. Discriminant $D_6 > 0$ everywhere, and so we conclude that in the absence of resonances equilibrium E_2, when linearly stable, is also Lyapunov stable.

V. Resonant Cases

Arnold's theorem implicitly assumes that Hamiltonian $\mathcal{H}_2$ is not resonant; in other words, that there is no rational commensurability among frequencies ω_1 and ω_2; otherwise, the normal form would not be as it appears to be in the theorem.

One of the main difficulties we face when dealing with a resonance of type $m{:}n$, when $\omega_1 = \Omega m$, $\omega_2 = \Omega n$ with Ω a frequency, and m, n coprime integers, is the presence of zero divisors when we try to obtain the normal form in Poincaré variables because the Lie derivative is $L_2(-) = m\partial(-)/\partial\phi_1 - n\partial(-)/\partial\phi_2$. However, we can circumvent this difficulty by means of the so-called extended Lissajous variables.[9] This set of canonical variables is defined by the symplectic transformation

$$\Psi_1 = m\Phi_1 + n\Phi_2, \qquad \phi_1 = \psi_1/2m + \psi_2/2n$$

$$\Psi_2 = m\Phi_1 - n\Phi_2, \qquad \phi_2 = \psi_1/2m - \psi_2/2n \tag{23}$$

In this set of variables, the zero order of Hamiltonian (9) is $\mathcal{H}_2 = \Omega\Psi_2$, and the Lie derivative is $L_2(-) = \Omega\partial(-)/\partial\psi_2$; thus, we can carry out the normalization in these variables avoiding the problem of zero divisors. Taking this into account, a function $F(\psi_1, \psi_2, \Psi_1, \Psi_2) \in \ker L_2$ if $\partial F/\partial\psi_2 = 0$, that is, if it does not contain the angle ψ_2.

For this Hamiltonian $\mathcal{H}_2$, it was proved[18] that there are four invariants:

$$M_1 = \tfrac{1}{2}\Psi_1$$
$$S = 2^{-(m+n)/2}(\Psi_1 - \Psi_2)^{m/2}(\Psi_1 + \Psi_2)^{n/2}\sin 2mn\psi_1$$
$$M_2 = \tfrac{1}{2}\Psi_2$$
$$C = 2^{-(m+n)/2}(\Psi_1 - \Psi_2)^{m/2}(\Psi_1 + \Psi_2)^{n/2}\cos 2mn\psi_1 \qquad (24)$$

that satisfy the relation

$$C^2 + S^2 = (M_1 + M_2)^n (M_1 - M_2)^m \qquad (25)$$

where $M_1 \geq |M_2|$.

Because the normalized Hamiltonian contains only the variables (ψ_1, Ψ_1, Ψ_2), it is possible to express it in function of the invariants (M_1, C, S) and the integral M_2 so that each term is

$$\mathcal{H}_s = \sum_{2(a_1+a_2)+(m+n)(a_3+a_4)=s} a_{a_1a_2a_3a_4} M_1^{a_1} M_2^{a_2} C^{a_3} S^{a_4} \qquad (26)$$

With the preceding invariants, the phase flow will be the level contours of the normalized Hamiltonian on the surface of revolution (25). For further details, see the work of Elipe.[18]

Noting that the change $(\Phi_1, \Phi_2) \mapsto (\omega_2, \omega_1)$ made in Arnold's theorem is equivalent to replacement $M_2 \mapsto 0$, Elipe and coworkers[10,19] proved that on the manifold $M_2 = 0$, if the normalized Hamiltonian $\mathcal{H}(M_1, C, S; 0)$ and the surface $C^2 + S^2 = M_1^{m+n}$ intersect transversally at the origin, the equilibrium is unstable. If the two surfaces have the origin as the unique common point, the equilibrium is stable.

A. Resonance 2:1

Let us first consider resonance 2:1, $\omega_1 = 2\omega_2 = 2\Omega$, which is a resonance of order 3 $(= m + n)$. Using Eq. (26) with $s = 3$, we see the intersection of the two surfaces $\mathcal{H}_3 = a_{0010}C + a_{0001}S$ and $C^2 + S^2 = M_1^3$, which after a rotation about axis M_1 can be

$$\left.\begin{aligned} \mathcal{H}_3 &= b_{0010}C \\ C^2 + S^2 &= M_1^3 \end{aligned}\right\}$$

If $b_{0010} \neq 0$, these surfaces intersect transversally, and consequently the equilibrium is unstable. On the contrary, if $b_{0010} = 0$ the stability has not yet been decided, and it is necessary to go to higher orders in the normalization.

From our computations we plot this coefficient b_{0010} vs α (Fig. 4). We see that $b_{0010} \neq 0$ except at value 0.982359 in which it is null. Thus, except for this point the equilibrium at resonance 2:1 is unstable. The set of values (α, β) at which the frequencies are in resonance 2:1 is plotted as the thin solid line in Fig. 2. We note that this curve and the one corresponding to the one at which $D_4 = 0$ cut each other, which would require the computing of D_6 to decide the stability at this particular case.

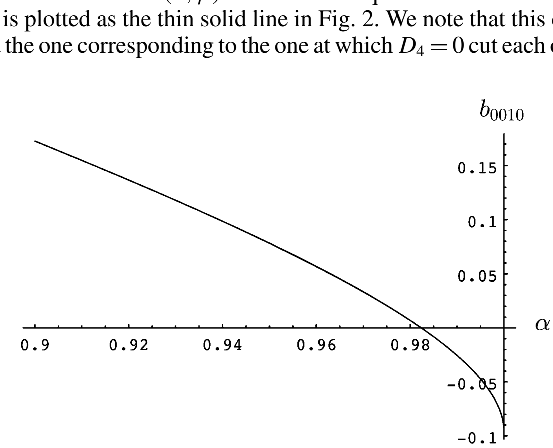

Fig. 4 Evolution of coefficient b_{0010} vs α for resonance 2:1. Coefficient $b_{0010} \neq 0$, except at one point at which the stability is not decided; thus, the equilibrium is Lyapunov unstable everywhere except, perhaps, at this point.

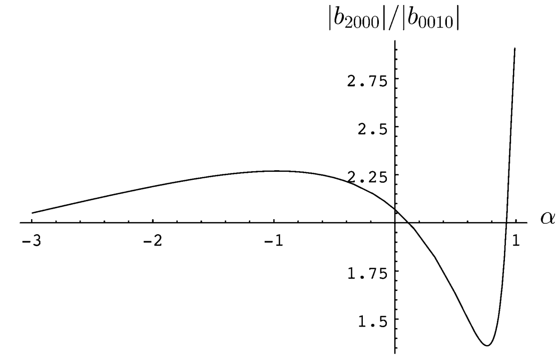

Fig. 5 Evolution of $|b_{2000}|/|b_{0010}|$ vs α for resonance 3:1. We see that $|b_{2000}| > |b_{0010}|$, thus, the equilibrium is Lyapunov stable everywhere.

B. Resonance 3:1

Let us consider now the fourth-order resonance 3:1, that is, $\omega_1 = 3\omega_2 = 3\Omega$; thus, according to Eq. (26),

$$\mathcal{H}_4 = a_{2000}M_1^2 + a_{1100}M_1M_2 + a_{0200}M_2^2 + a_{0010}C + a_{0001}S$$

Along the manifold $M_2 = 0$, the two surfaces are $\mathcal{H}_4 = a_{2000}M_1^2 + a_{0010}C + a_{0001}S$, and $C^2 + S^2 = M_1^4$.

Again, after an appropriate rotation about the M_1 axis, they are converted into

$$\left.\begin{aligned} \mathcal{H}_4 &= b_{2000}M_1^2 + b_{0010}C \\ S^2 + C^2 &= M_1^4 \end{aligned}\right\}$$

Then, a projection onto the plane $S = 0$ and taking $\mathcal{H}_4 = 0$ yields on

$$\left.\begin{aligned} C &= -\frac{b_{2000}}{b_{0010}}M_1^2 \\ C &= \pm M_1^2 \end{aligned}\right\}$$

$$\Longrightarrow \left.\begin{array}{l} \text{If } |b_{0010}| < |b_{2000}| \mapsto \nexists \text{ cuts, then STABLE.} \\ \text{If } |b_{0010}| > |b_{2000}| \mapsto \exists \text{ cuts, then UNSTABLE.} \end{array}\right\}$$

In Fig. 5 we plotted $|b_{2000}|/|b_{0010}|$ vs α. This quotient is greater than 1 in the whole domain; hence, we can conclude that for resonance 3:1 the equilibrium is stable everywhere.

VI. Conclusions

Stationary points are very interesting points to place orbiters around a planet. In the case of the Earth, geostationary points have been widely used; however, space missions around other celestial bodies make it necessary to analyze them for these bodies. In this paper, we determine the existence and their Lyapunov stability (even for resonances) depending on two parameters, which essentially are the second order and degree harmonics of the potential expansions. Because the analysis has been made symbolically, for missions to other celestial bodies it would be a matter of computing the stability parameters α and β by using potential harmonic coefficients and checking in which region of the stability diagram the parameters are to determine the stability of the stationary points.

Acknowledgments

Supported by the Spanish Ministry of Science and Technology (Projects # ESP2002-02329, AYA 2004-03298). The authors appear in alphabetical order.

References

[1] Soop, E. M., *Handbook of Geostationary Orbits*, Space Technology Library Series, Kluwer Academic, Dordrecht, The Netherlands, 1994, Chap. 1.

[2]Blitzer, L., Boughton, E. M., Kang, G., and Page, R. M., "Effect of Ellipticity of the Equator on 24-Hour Nearly Circular Satellite Orbits," *Journal of Geophysical Research*, Vol. 67, No. 1, 1962, pp. 329–335.

[3]Musen, P., and Bailie, A. E., "On the Motion of a 24-Hour Satellite," *Journal of Geophysical Research*, Vol. 67, March 1962, pp. 1123–1132.

[4]Morando, B., "Orbites de Résonance des Satellites de 24 Heures," *Bulletin Astronomique*, Vol. 24, No. 1, 1963, pp. 47–67.

[5]Deprit, A., and López-Moratalla, T., "Estabilidad Orbital de Satélites Estacionarios," *Revista Matemática Universidad Complutense*, Vol. 9, No. 2, 1996, pp. 311–333.

[6]López-Moratalla, T., "Estabilidad Orbital de Satélites Estacionarios," Ph. D. Dissertation, Univ. de Zaragoza, Spain, *Boletín ROA*, Vol. 97, No. 5, 1997 (in Spanish).

[7]Deprit, A., and Deprit-Bartholomé, A., "Stability of the Triangular Lagrangian Points," *Astronomical Journal*, Vol. 72, No. 2, 1967, pp. 173–179.

[8]Arnold, V. I., "The Stability of the Equilibrium Position of a Hamiltonian System of Ordinary Differential Equations in the General Elliptic Case," *Soviet Mathematics Doklady*, Vol. 2, 1961, pp. 247–249.

[9]Elipe, A., and Deprit, A., "Oscillators in Resonance," *Mechanics Research Communications*, Vol. 26, No. 6, 1999, pp. 635–640.

[10]Elipe, A., Lanchares, V., López-Moratalla, T., and Riaguas, A., "Nonlinear Stability in Resonant Cases: A Geometrical Approach," *Journal of Nonlinear Sciences*, Vol. 11, No. 3, 2001, pp. 211–222.

[11]Heiskanen, W. A., and Moritz, H., *Physical Geodesy,* Freeman Co., San Francisco, 1967, pp. 46–125.

[12]Szebehely, V., *Theory of Orbits,* Academic Press, New York, 1967, pp. 59–61.

[13]Howard, J., "Spectral Stability of Relative Equilibria," *Celestial Mechanics,* Vol. 48, No. 3, 1990, pp. 267–288.

[14]Meirovitch, L., *Methods of Analytical Dynamics,* 2nd ed., McGraw–Hill, New York, 1988, Chap. 6.

[15]Laub, A., and Meyer, K., "Canonical Forms for Symplectic and Hamiltonian Matrices," *Celestial Mechanics,* Vol. 9, No. 2, 1974, pp. 213–238.

[16]Deprit, A., "Motion in the Vicinity of the Triangular Libration Centers," Lectures in Applied Mathematics (American Mathematical Society) *Space Mathematics*, Vol. 6, 1966, Part II, pp. 1–30.

[17]Deprit, A., "Canonical Transformations Depending on a Small Parameter," *Celestial Mechanics,* Vol. 1, No. 1, 1969, pp. 12–30.

[18]Elipe, A., "Complete Reduction of Oscillators in Resonance p:q," *Physical Reviews E*, Vol. 61, No. 6, 2000, pp. 6477–6488.

[19]Elipe, A., Lanchares, V., and Pascual, A. I., "On the Stability of Equilibria in Two Degrees of Freedom Hamiltonian Systems," *Journal of Nonlinear Sciences*, Vol. 15, No. 5, 2005, pp. 305–319.